Matching heavy-quark fields in QCD and HQET at three loops

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Abstract

The relation between the heavy-quark field in QCD and the corresponding field in HQET is derived up to three loops, and to all orders in the large-β_0 limit. The corresponding relation between the QED electron field and the Bloch–Nordsieck one is gauge invariant to all orders. We also prove that the $\overline{\text{MS}}$ anomalous dimension of the QED electron field depends on the gauge parameter only at one loop.

QCD problems with a single heavy quark Q can be treated in a simpler effective theory — HQET, if there exists a 4-velocity v such that the heavy-quark momentum is $p = mv + k$ (m is the on-shell mass) and the characteristic residual momentum is small: $k \ll m$. QCD operators can be written as series in $1/m$ via HQET operators; the coefficients in these series are determined by matching on-shell matrix elements in both theories. For example, the heavy–light quark currents have been considered at the leading (zeroth) order in $1/m$ up to three-loop accuracy [1,2].

Here we shall consider the heavy-quark field Q. Though its matrix elements are not directly observable, its matching coefficient can have some applications. For example, it is not possible to simulate heavy quarks on a lattice directly, because at present we cannot have lattice spacings $a \ll 1/m$. On the other hand, simulating HQET on a lattice only requires $a \ll 1/\Lambda_{\overline{\text{MS}}}$. It is possible to obtain the HQET heavy-quark propagator in the Landau gauge from such simulations. Then, if we know the matching coefficient, we can reconstruct a fundamental QCD quantity — the heavy-quark propagator as a function of x.

At the tree level, Q is related to the corresponding HQET field Q_v (satisfying $\rlap/v Q_v = Q_v$) by [3,4,5,6]

$$Q(x) = e^{-imv\cdot x}\left(1 + \frac{i\rlap/D_\perp}{2m} + \cdots\right) Q_v(x)\,, \quad D_\perp^\mu = D^\mu - v^\mu v\cdot D\,. \qquad (1)$$

The matrix elements of the bare fields between the on-shell quark with momentum $p = mv + k$ and the vacuum in both theories are given by the on-shell wave-function renormalization constants:

$$<0|Q_0|Q(p)> = \left(Z_Q^{\text{os}}\right)^{1/2} u(p)\,, \quad <0|Q_{v0}|Q(p)> = \left(\tilde{Z}_Q^{\text{os}}\right)^{1/2} u_v(k) \tag{2}$$

(HQET renormalization constants are denoted by $\tilde{Z}$). The Dirac spinors are related by the Foldy–Wouthuysen transformation

$$u(mv+k) = \left[1 + \frac{\not{k}}{2m} + \mathcal{O}\left(\frac{k^2}{m^2}\right)\right] u_v(k)\,.$$

Therefore, the bare fields are related by

$$Q_0(x) = e^{-imv\cdot x}\left[z_0^{1/2}\left(1 + \frac{i\not{D}_\perp}{2m}\right) Q_{v0}(x) + \mathcal{O}\left(\frac{1}{m^2}\right)\right]\,, \tag{3}$$

where the bare matching coefficient is

$$z_0 = \frac{Z_Q^{\text{os}}(g_0^{(n_l+1)}, a_0^{(n_l+1)})}{\tilde{Z}_Q^{\text{os}}(g_0^{(n_l)}, a_0^{(n_l)})} \tag{4}$$

(we use the covariant gauge: the gauge-fixing term in the Lagrangian is $-(\partial_\mu A_0^{a\mu})/(2a_0)$, and the free gluon propagator is $(-i/p^2)(g_{\mu\nu} - (1 - a_0)p_\mu p_\nu/p^2)$; the number of flavours in QCD is $n_f = n_l + 1$). The $\mathcal{O}(1/m)$ matching coefficient in (3) is equal to the leading one, z_0; this reflexes the reparametrization invariance [7]. The $\overline{\text{MS}}$ renormalized fields are related by the formula similar to (3), with the renormalized decoupling coefficient

$$z(\mu) = \frac{\tilde{Z}_Q(\alpha_s^{(n_l)}(\mu), a^{(n_l)}(\mu))}{Z_Q(\alpha_s^{(n_l+1)}(\mu), a^{(n_l+1)}(\mu))} z_0\,. \tag{5}$$

If there are no massive flavours except Q, then $\tilde{Z}_Q^{\text{os}} = 1$ because all loop corrections are scale-free. The QCD on-shell renormalization constant Z_Q^{os} contains the single scale m in this case; it has been calculated [8] up to three loops. The three-loop $\overline{\text{MS}}$ anomalous dimensions of Q_v [8,9] and Q [10,11] are also known. We have to express all three quantities $Z_Q^{\text{os}}(g_0^{(n_l+1)}, a_0^{(n_l+1)})$, $Z_Q(\alpha_s^{(n_l+1)}(\mu), a^{(n_l+1)}(\mu))$, $\tilde{Z}_Q(\alpha_s^{(n_l)}(\mu), a^{(n_l)}(\mu))$ via the same variables, say, $\alpha_s^{(n_l)}(\mu)$, $a^{(n_l)}(\mu)$, see [12]. An explicit formula expressing $\alpha_s^{(n_l+1)}(\mu)$ via $\alpha_s^{(n_l)}(\mu)$ and $L = 2\log(\mu/m)$ (m is the on-shell mass) can be found in [13]. The corre-

sponding relation between $a^{(n_l+1)}(\mu)$ and $a^{(n_l)}(\mu)$ is

$$\begin{aligned}\frac{a^{(n_l+1)}(\mu)}{a^{(n_l)}(\mu)} = 1 &- \left[\frac{4}{3}L + \frac{6L^2+\pi^2}{9}\varepsilon + \frac{2L^3+\pi^2 L - 4\zeta_3}{9}\varepsilon^2 + \cdots\right] T_F \frac{\alpha_s^{(n_l)}(\mu)}{4\pi} \\ &- \Bigg[C_A L^2 + (4C_F + 5C_A)\, L + 15 C_F - \frac{13}{12}C_A \\ &\quad + \Bigg(C_A L^3 + (4C_F+5C_A)\, L^2 + \left(30 C_F + \frac{\pi^2-13}{6}C_A\right) L \\ &\qquad + \left(\frac{\pi^2}{3}+\frac{31}{2}\right) C_F + \frac{5\pi^2+169}{12}C_A\Bigg)\varepsilon + \cdots\Bigg] T_F \left(\frac{\alpha_s^{(n_l)}(\mu)}{4\pi}\right)^2 + \cdots\end{aligned} \tag{6}$$

Our main result is the renormalized matching coefficient

$$\begin{aligned} z(\mu) =& 1 - (3L+4) C_F \frac{\alpha_s^{(n_l)}(\mu)}{4\pi} + \left(z_{22}L^2 + z_{21}L + z_{20}\right) C_F \left(\frac{\alpha_s^{(n_l)}(\mu)}{4\pi}\right)^2 \\ &+ \left(z_{33}L^3 + z_{32}L^2 + z_{31}L + z_{30}\right) C_F \left(\frac{\alpha_s^{(n_l)}(\mu)}{4\pi}\right)^3 + \cdots \end{aligned} \tag{7}$$

where

$$\begin{aligned} z_{22} =& \frac{9}{2}C_F - \frac{11}{2}C_A + 2T_F n_l\,, \\ z_{21} =& \frac{27}{2}C_F - \frac{215}{6}C_A + \frac{38}{3}T_F n_l + 2T_F\,, \\ z_{20} =& \left(16\pi^2\log 2 - 24\zeta_3 - 13\pi^2 + \frac{241}{8}\right) C_F \\ &+ \left(-8\pi^2 \log 2 + 12\zeta_3 + 5\pi^2 - \frac{1705}{24}\right) C_A \\ &+ \left(\frac{4}{3}\pi^2 + \frac{113}{6}\right) T_F n_l + \left(-\frac{16}{3}\pi^2 + \frac{947}{18}\right) T_F\,, \\ z_{33} =& -\frac{9}{2}C_F^2 + \frac{33}{2}C_F C_A - \frac{121}{9}C_A^2 - 6C_F T_F n_l \\ &+ \frac{88}{9}C_A T_F n_l + \frac{1}{3}a^{(n_l)}(\mu) C_A T_F - \frac{16}{9}T_F^2 n_l^2\,, \\ z_{32} =& -\frac{45}{2}C_F^2 + 135 C_F C_A - \frac{2671}{18}C_A^2 - 42 C_F T_F n_l - 4C_F T_F \\ &+ \frac{938}{9}C_A T_F n_l - \left(\frac{13}{6}a^{(n_l)}(\mu) + 1\right) C_A T_F - \frac{152}{9}T_F^2 n_l^2 + \frac{8}{3}T_F^2\,, \\ z_{31} =& \left[-48\pi^2\log 2 + 72\zeta_3 + 39\pi^2 - \frac{783}{8}\right] C_F^2 \\ &+ \left[\frac{424}{3}\pi^2 \log 2 - 224\zeta_3 - \frac{331}{3}\pi^2 + \frac{13307}{24}\right] C_F C_A \end{aligned}$$

$$
\begin{aligned}
&+ \left[\left(-\frac{2}{45}\pi^4 + \frac{9}{4}\zeta_3 + \frac{1}{4}\right)a^{(n_l)}(\mu) \right.\\
&\quad \left. - \frac{176}{3}\pi^2\log 2 + \frac{325}{4}\zeta_3 - \frac{2}{15}\pi^4 + \frac{110}{3}\pi^2 - \frac{73981}{108}\right]C_A^2 \\
&+ \left(-\frac{128}{3}\pi^2\log 2 + 16\zeta_3 + \frac{92}{3}\pi^2 - \frac{613}{6}\right)C_F T_F n_l + \left(16\pi^2 - \frac{1013}{6}\right)C_F T_F \\
&+ \left(\frac{64}{3}\pi^2\log 2 + 16\zeta_3 - \frac{32}{9}\pi^2 + \frac{10816}{27}\right)C_A T_F n_l \\
&+ \left(\frac{121}{18}a^{(n_l)}(\mu) - \frac{352}{9}\pi^2 + \frac{11278}{27}\right)C_A T_F \\
&- \left(\frac{32}{9}\pi^2 + \frac{1336}{27}\right)T_F^2 n_l^2 + \left(\frac{128}{9}\pi^2 - \frac{3908}{27}\right)T_F^2 n_l - \frac{20}{9}T_F^2\,,
\end{aligned}
$$

$$
\begin{aligned}
z_{30} =& \left[-1792 a_4 - \frac{224}{3}\log^4 2 + 96\pi^2\log^2 2 + \frac{3568}{3}\pi^2\log 2 - 20\zeta_5 \right.\\
&\quad \left. + 8\pi^2\zeta_3 - 1256\zeta_3 - \frac{76}{15}\pi^4 - \frac{4801}{9}\pi^2 - \frac{3023}{12}\right]C_F^2 \\
&+ \left[-\frac{32}{3}a_4 - \frac{4}{9}\log^4 2 - \frac{1448}{9}\pi^2\log^2 2 - \frac{2752}{9}\pi^2\log 2 + 580\zeta_5 \right.\\
&\quad \left. - 180\pi^2\zeta_3 - \frac{2312}{3}\zeta_3 + \frac{6697}{270}\pi^4 + \frac{2137}{9}\pi^2 + \frac{24131}{72}\right]C_F C_A \\
&+ \left[\left(-\frac{7}{6}\zeta_5 - \frac{4}{9}\pi^2\zeta_3 + \frac{13}{4}\zeta_3 - \frac{17}{432}\pi^4 + \frac{1}{4}\pi^2 + \frac{13}{12}\right)a^{(n_l)}(\mu) \right.\\
&\quad + \frac{1360}{3}a_4 + \frac{170}{9}\log^4 2 + \frac{508}{9}\pi^2\log^2 2 - \frac{1300}{9}\pi^2\log 2 - \frac{787}{2}\zeta_5 \\
&\quad \left. + \frac{340}{3}\pi^2\zeta_3 + \frac{23311}{36}\zeta_3 - \frac{20429}{2160}\pi^4 - \frac{8705}{108}\pi^2 - \frac{1656817}{1944}\right]C_A^2 \\
&+ \left[\frac{1024}{3}a_4 + \frac{128}{9}\log^4 2 + \frac{256}{9}\pi^2\log^2 2 - \frac{1504}{9}\pi^2\log 2 \right.\\
&\quad \left. + \frac{1096}{3}\zeta_3 - \frac{916}{135}\pi^4 + \frac{904}{9}\pi^2 + \frac{1120}{9}\right]C_F T_F n_l \\
&+ \left[768 a_4 + 32\log^4 2 - 32\pi^2\log^2 2 + \frac{1088}{9}\pi^2\log 2 \right.\\
&\quad \left. + \frac{466}{9}\zeta_3 + \frac{124}{45}\pi^4 - \frac{8848}{81}\pi^2 - \frac{16811}{54}\right]C_F T_F \\
&+ \left[-\frac{512}{3}a_4 - \frac{64}{9}\log^4 2 - \frac{128}{9}\pi^2\log^2 2 + \frac{752}{9}\pi^2\log 2 \right.\\
&\quad \left. - \frac{280}{9}\zeta_3 + \frac{152}{135}\pi^4 + \frac{52}{3}\pi^2 + \frac{111791}{243}\right]C_A T_F n_l
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\left(\frac{8}{3}\zeta_3-\frac{2461}{108}\right)a^{(n_l)}(\mu)-512a_4-\frac{64}{3}\log^4 2+\frac{64}{3}\pi^2\log^2 2+\frac{5120}{9}\pi^2\log 2\right.\\
&\qquad\left.-60\zeta_5+\frac{44}{3}\pi^2\zeta_3-\frac{2837}{9}\zeta_3-\frac{136}{45}\pi^4-\frac{36268}{81}\pi^2+\frac{100627}{81}\right]C_A T_F\\
&-\left(\frac{224}{9}\zeta_3+\frac{304}{27}\pi^2+\frac{11534}{243}\right)T_F^2 n_l^2+\left(\frac{208}{9}\pi^2-\frac{18884}{81}\right)T_F^2 n_l\\
&+\left(\frac{448}{3}\zeta_3+\frac{128}{45}\pi^2-\frac{16850}{81}\right)T_F^2
\end{aligned}
$$

(here $a_4 = \mathrm{Li}_4(1/2)$). Gauge dependence first appears at three loops, as in Z_Q^{os} [8]. The requirement of finiteness of the renormalized matching coefficient (5) at $\varepsilon \to 0$ has allowed the authors of [8] to extract $\tilde{Z}_Q$ from their result for Z_Q^{os}.

It would not be too difficult to take into account a lighter massive flavour, say, $m_c \neq 0$ in b-quark HQET. $\tilde{Z}_Q^{\mathrm{os}}$ is no longer equal to 1, but is known at three loops [14]; Z_Q^{os} contains two scales, and is a non-trivial function of m_c/m_b [15]. Both $\tilde{Z}_Q^{\mathrm{os}}$ and Z_Q^{os} have no smooth limit at $m_c \to 0$, but the discontinuity cancels in the ratio (4).

Now let's consider $z(\mu)$ in the large-β_0 limit (see Chapter 8 in [16] for a pedagogical introduction):

$$
z(\mu) = 1 + \int_0^\beta \frac{d\beta}{\beta}\left(\frac{\gamma(\beta)}{2\beta}-\frac{\gamma_0}{2\beta_0}\right) + \frac{1}{\beta_0}\int_0^\infty du\, e^{-u/\beta} S(u) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \quad (8)
$$

where $\beta = \beta_0\alpha_s/(4\pi)$, $\gamma = \gamma_0\alpha_s/(4\pi) + \cdots$ (differences of n_l-flavour and (n_l+1)-flavour quantities can be neglected at the $1/\beta_0$ order). The difference of the QCD and HQET anomalous dimensions $\gamma = \gamma_Q - \tilde{\gamma}_Q$ and the Borel image $S(u)$ can be expressed as

$$
\gamma(\beta) = -2\frac{\beta}{\beta_0}F(-\beta,0)\,, \quad S(u) = \frac{F(0,u)-F(0,0)}{u}\,, \quad (9)
$$

where the function $F(\varepsilon,u)$ has been calculated in [1] (see also [16]):

$$
\begin{aligned}
F(\varepsilon,u) ={}& -2C_F\left(\frac{\mu}{m}\right)^{2u} e^{\gamma_E\varepsilon}\frac{\Gamma(1+u)\Gamma(1-2u)}{\Gamma(3-u-\varepsilon)}D(\varepsilon)^{u/\varepsilon-1}\\
&\times(3-2\varepsilon)(1-u)(1+u-\varepsilon)\,, \qquad (10)\\
D(\varepsilon) ={}& 6e^{\gamma_E\varepsilon}\Gamma(1+\varepsilon)B(2-\varepsilon,2-\varepsilon) = 1+\frac{5}{3}\varepsilon+\cdots
\end{aligned}
$$

The anomalous dimension difference [1] is gauge invariant at this order:

$$
\gamma(\beta) = 2C_F\frac{\beta}{\beta_0}\frac{(1+\beta)(1+\frac{2}{3}\beta)}{B(2+\beta,2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)}\,; \quad (11)
$$

the Borel image is [17,16]

$$S(u) = -6C_F \left[e^{(L+5/3)u} \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} (1-u^2) - \frac{1}{2u} \right] . \tag{12}$$

This Borel image has infrared renormalon poles at each positive half-integer u and at $u = 2$. Therefore, the integral in (8) is not well defined. Comparing its residue at the leading pole $u = 1/2$ with the residue of the static-quark self-energy at its ultraviolet pole $u = 1/2$ [18], we can express the renormalon ambiguity of $z(\mu)$ as

$$\Delta z(\mu) = \frac{3}{2} \frac{\Delta\bar{\Lambda}}{m} \tag{13}$$

($\bar{\Lambda}$ is the ground-state meson residual energy). This ambiguity is compensated in physical matrix elements by ultraviolet renormalon ambiguities in the leading $1/m$ correction (matrix elements of both local and bilocal dimension-5/2 operators), see [17].

The matching coefficient is gauge invariant at the order $1/\beta_0$. Expanding $\gamma(\beta)$ and $S(u)$ and integrating, we obtain

$$\begin{aligned} z(\mu) = 1 - C_F \frac{\alpha_s(\mu)}{4\pi} \biggl\{ & 3L + 4 + \left[\frac{3}{2} L^2 + \frac{19}{2} L + \pi^2 + \frac{113}{8} \right] \frac{\beta_0 \alpha_s(\mu)}{4\pi} \\ & + \left[L^3 + \frac{19}{2} L^2 + \left(2\pi^2 + \frac{167}{6} \right) L + 14\zeta_3 + \frac{19}{3}\pi^2 + \frac{5767}{216} \right] \left(\frac{\beta_0 \alpha_s(\mu)}{4\pi} \right)^2 \\ & + \left[\frac{3}{4} L^4 + \frac{19}{2} L^3 + \left(3\pi^2 + \frac{167}{4} \right) L^2 + \left(36\zeta_3 + 19\pi^2 + \frac{2903}{36} \right) L \right. \\ & \left. + \frac{71}{40}\pi^4 + \frac{467}{4}\zeta_3 + \frac{167}{6}\pi^2 + \frac{103933}{1728} \right] \left(\frac{\beta_0 \alpha_s(\mu)}{4\pi} \right)^3 + \cdots \biggr\} . \end{aligned} \tag{14}$$

Thus we have confirmed the contributions with the highest power of n_l in each term in (7), and predicted such a contribution at α_s^4.

Numerically, the matching coefficient (7) in the Landau gauge at $\mu = m$ and $n_l = 4$ can be written as

$$\begin{aligned} z(m) = 1 - \frac{4}{3} \frac{\alpha_s^{(4)}(m)}{\pi} & - (1.9996\beta_0 - 4.5421) \left(\frac{\alpha_s^{(4)}(m)}{\pi} \right)^2 \\ & - (2.2091\beta_0^2 + 5.1153\beta_0 - 61.5397) \left(\frac{\alpha_s^{(4)}(m)}{\pi} \right)^3 \\ & - (3.3755\beta_0^3 + \cdots) \left(\frac{\alpha_s^{(4)}(m)}{\pi} \right)^4 + \cdots \end{aligned}$$

$$
\begin{aligned}
&= 1 - \frac{4}{3}\frac{\alpha_s^{(4)}(m)}{\pi} - (16.6629 - 4.5421)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^2 \\
&\quad - (153.4076 + 42.6271 - 61.5397)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^3 \\
&\quad - (1953.4013 + \cdots)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^4 + \cdots \qquad (15) \\
&= 1 - \frac{4}{3}\frac{\alpha_s^{(4)}(m)}{\pi} - 12.1208\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^2 - 134.4950\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^3 \\
&\quad - (1953.4013 + \cdots)\left(\frac{\alpha_s^{(4)}(m)}{\pi}\right)^4 + \cdots
\end{aligned}
$$

(β_0 is for $n_l = 4$ flavours). Naive nonabelianization [1] works rather well at two and three loops (in the latter case the $\mathcal{O}(\beta_0)$ and $\mathcal{O}(1)$ terms partially compensate each other, similarly to [2]). Therefore, we can expect that the estimate of the α_s^4 term is also reasonably good. Numerical convergence of the series is very poor; this is related to the infrared renormalon at $u = 1/2$.

Now let us consider the relation between the $\overline{\mathrm{MS}}$ renormalized electron field in QED and the Bloch–Nordsieck electron field [19]. The bare matching coefficient $z_0 = Z_\psi^{\mathrm{os}}$ is gauge invariant to all orders [20,21,8]. In the Bloch-Nordsieck model, due to exponentiation [22], $\log \tilde{Z}_\psi = (3 - a^{(0)})\alpha^{(0)}/(4\pi\varepsilon)$ (where the 0-flavour $\alpha^{(0)}$ is equal to the on-shell $\alpha \approx 1/137$). In the full QED, $\log Z_\psi = -a^{(1)}\alpha^{(1)}/(4\pi\varepsilon) +$ (gauge-invariant higher terms), see the Appendix. The gauge dependence cancels in $\log(\tilde{Z}_\psi/Z_\psi)$ because of the QED decoupling relation $a^{(1)}\alpha^{(1)} = a^{(0)}\alpha^{(0)}$. Therefore, the renormalized matching coefficient $z(\mu)$ in QED is gauge invariant to all orders. We obtain

$$
\begin{aligned}
z(\mu) =&1 - (3L + 4)\frac{\alpha}{4\pi} \\
&+ \left(\frac{9}{2}L^2 + \frac{31}{2}L + 16\pi^2\log 2 - 24\zeta_3 - \frac{55}{3}\pi^2 + \frac{5957}{72}\right)\left(\frac{\alpha}{4\pi}\right)^2 \\
&- \left[\frac{9}{2}L^3 + \frac{143}{6}L^2 + \left(48\pi^2\log 2 - 72\zeta_3 - 55\pi^2 + \frac{19363}{72}\right)L \right. \\
&\quad + 1024a_4 + \frac{128}{3}\log^4 2 - 64\pi^2\log^2 2 - \frac{11792}{9}\pi^2\log 2 + 20\zeta_5 \\
&\quad \left. - 8\pi^2\zeta_3 + \frac{9494}{9}\zeta_3 + \frac{104}{45}\pi^4 + \frac{259133}{405}\pi^2 + \frac{249887}{324}\right]\left(\frac{\alpha}{4\pi}\right)^3 + \cdots
\end{aligned}
\qquad (16)
$$

In conclusion: we have derived the QCD/HQET matching coefficient for the heavy-quark field with three-loop accuracy (7), and the all-orders result in the large-β_0 limit. The corresponding QED coefficient (16) is gauge invariant.

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Appendix. Gauge dependence of Z_ψ and γ_ψ in QED.

The electron propagator $S(x)$ is related to the Landau-gauge propagator $S_L(x)$ ($a_0 = 0$) by the formula [20]

$$S(x) = S_L(x) e^{-ie_0^2(\Delta(x)-\Delta(0))}\,, \tag{17}$$

where $\Delta(x)$ is the Fourier image of $\Delta(k) = a_0/(k^2)^2$, and $\Delta(0) = 0$ in dimensional regularization. The electron field renormalization does not depend on its mass. For simplicity, we shall consider the massless electron, whose propagator has a single Dirac structure:

$$S(x) = S_0(x) e^{\sigma(x)}\,, \tag{18}$$

where $S_0(x)$ is the d-dimensional free massless electron propagator. Then

$$\sigma(x) = \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4}\right)^\varepsilon \Gamma(-\varepsilon)\,, \tag{19}$$

where the Landau-gauge σ_L starts from e_0^4. Re-expressing σ via renormalized quantities we have

$$\sigma(x) = \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4}\right)^\varepsilon e^{\gamma_E \varepsilon} \Gamma(-\varepsilon)\,. \tag{20}$$

This must be equal to $\log Z_\psi + \sigma_r$, where $\log Z_\psi(\alpha(\mu), a(\mu))$ contains only negative powers of ε, and the renormalized σ_r — only non-negative. Therefore,

$$\log Z_\psi(\alpha, a) = \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon}\,. \tag{21}$$

In QED $d\log(a(\mu)\alpha(\mu))/d\log\mu = -2\varepsilon$ exactly, and [1]

$$\gamma_\psi(\alpha, a) = 2a \frac{\alpha}{4\pi} + \gamma_L(\alpha)\,, \tag{22}$$

where the Landau-gauge $\gamma_L(\alpha)$ starts from α^2.

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[1] I was informed by D.J. Broadhurst and D.V. Shirkov that this result has been proved in some Russian article in the second half of 50s. I am grateful to them for discussing this question; unfortunately, I was unable to find this article.

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